Fracto-Morphodynamics

Navjot Singh

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Abstract

In this work the connection between curvature, the emergence of the universe, and the process of morphogenesis is investigated in detail. We have come a long way from general relativity using differential geometry, which describes the curvature of space-time as an emergent phenomena, to the more recent theory of loop quantum gravity using non-geometric objects such as holonomy, fluxes, spin, and knotting. However, these theories are still only a model for space-time curvature, i.e., they explain the coarse-grained description of the structure of space-time and how this can account for the scale-variant behaviour and its dynamics. In this work, I will use the term morphogenesis to describe the emergence of structure in space-time fabric. The different levels of scale of phenomena can be described by their fractal analogues.

Keywords: fractomorphodynamics, morphogenesis, fractal dimensions, fractal space-time

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1 Introdcution

A unified description of gravity is a longstanding problem in physics. Since the early history of physics it has been proposed that space-time was discrete, a question that was taken up again in the 1980s when physicists started to take the nature of quantum gravity seriously. So far, all attempts to find a theory of everything have been plagued by the problem of singularities. While these theories are very successful at explaining physics on the largest scales of astrophysics, they cannot describe the physics on the smallest scale of atoms and elementary particles. One of the latest efforts to find a theory of everything uses a geometric approach to finding the quantum mechanics of gravity based on a discrete version of general relativity called loop quantum gravity. In these theories, gravity and quantum mechanics are unified using non-geometric objects such as holonomy, fluxes, spin, and knotting (see, e.g., [**1**]. However, all of these theories describe the emergent nature of the geometry of space, not space as a substance. To explain the full behavior of space, we need to provide an explanation of the internal behavior of space. Here I will offer a purely geometric description of this emergent process known as morphogenesis that describes the process of space-time developing its structure. This article explains the relationship between spacetime curvature, topology change, and the fractal nature of space. The key concept of this article is the geometric morphogenesis of space-time. This approach leads to a unified understanding of general relativity, and the standard model of particle physics as facets of the same phenomena. These new ideas replace the current idea of a space-time manifold of differential equations with a nonlinear space-time model that is not limited by the smooth and continuous approximation of general relativity, but can also be applied to quantum theories of gravity. Unlike many other approaches to quantum gravity, which are still using differential geometry in their theory, this new geometric approach starts with the assumption of an underlying fractal. With a fractal, you have different scales of smoothness, i.e., from coarse to fine. Based on the different scales of smoothness, you get a set of differential equations that will allow you to describe the degrees of curvature at varying scales. This description is carried out using a morphogenetic potential, which gives the geometric curvature of spacetime.

2 Geometric Morphogensis

$$\begin{tikzcd}
& \Lambda_{\delta=4}\left(\psi^{C}(4)\right) \arrow[r, "\mathrm{Hol}\times -\mathrm{Id}"] \arrow[d] & \Lambda_{\delta=2}\left(\phi^{C}(4)\right) \arrow[d] \\
\phi^{C}_{O}(1,3) & \Lambda_{\delta=4}\left(\mathbb{R}^{1,3}\right) \times \Lambda_{\delta=4}\left(\mathbb{R}^{3,1}\right) \arrow[l, "-\mathrm{Id}\times\mathrm{Hol}"'] \arrow[ur, "\mathrm{Hol}\times\mathrm{Hol}"] & \phi^{C}_{O}(0,4)
\end{tikzcd}$$

Figure 1: Morphogenesis of Space-time

Morphogenesis: It's defined as the formation of structure in spacetime as a function of the geometric morphogenetic potential f, and the external forcing or boundary conditions b. The geometric morphogenetic potential is an abstract vector field describing the structure formation that emerges. For example, if it is an attractor field, it has a stable space of equilibria. If it is a pseudo force field, it has no stable point of equilibria, but instead it has a fractal structure with fixed points and saddles. In the former case, the dynamics has an eigen spectrum with discrete eigenvalues. In the latter, the eigenvalues are dense on the real axis, which is consistent with self-similar fractals. In order to define the geometric morphogenetic potential as an abstract vector field, we first identify the n-dimensional morphogenetic potential space as

$$\Lambda_n = \underbrace{\left(\frac{\mathbb{R}}{\mathbb{Z}}\right) \times \cdots \times \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)}_{n\,\text{times}} \tag{1}$$

We then projectively identify each of the Λ_n's along the temporal coordinate, so that Λ_n becomes identified with $\mathbb{R}^n$. The end result is that we have a single infinite dimensional vector space whose basis is $\mathcal{B}_{n,i}$ where i indexes the dimension of the physical space, e.g.:

$$\mathcal{B}_{n,i} = \{\text{point}\, O^i_t(0);\ t \in \mathbb{R}\} \cup \tag{2}$$

$$\{\text{line}\, l^i_{n,t};\ l^i_{n,t} \text{ is a 1-cycle with period } \tau_{n,i}\} \cup \tag{3}$$

$$\{\text{surface}\, s^i_{n,t};\ s^i_{n,t} \text{ is a 2-cycle with period } \sigma_{n,i}\} \cup \tag{4}$$

$$\{\text{volume}\, v^i_{n,t};\ v^i_{n,t} \text{ is a 3-cycle with period } d_{n,i}\} \tag{5}$$

$$\ldots \tag{6}$$

The key idea in morphogenesis is that the field acting on a domain gives you not just a volume of that domain, but a certain type of shape of that volume. In addition, the field also specifies how fast that shape is changing. The general form of morphogenetic fields we consider is as follows:

$$\vec{\phi} = \Sigma_{k,m,l} \int_{D^k_{t_0,t_1}} \lambda^m_{k,l} \rho_{k,l} f_{k,m,l}\, dV \tag{7}$$

where $f_{k,m,l}$ is the geometric morphogenetic potential, $\rho_{k,l}$ is a measure density that we often choose to be a Dirac delta function, and $\lambda^m_{k,l}$ is a measure of the characteristic length-scale associated with the geometric morphogenetic potential $f_{k,m,l}$. In order to get a non-linear equation describing the structure of spacetime, we need to assume that the morphogenetic fields are constrained by the incompressibility condition:

$$\int_{D^k_{t_0,t_1}} \rho_{k,l}\, dV = 1 \tag{8}$$

which means that for a given domain $D^k_{t_0,t_1}$, at every instant in time, the volume associated with that domain is a constant, i.e.,

$$D^k_{t_0,t_1} = \bigcup_l D^{k,l}_{t_0,t_1} \tag{9}$$

$$\dot{D}^k_{t_0,t_1} = \bigcup_l \dot{D}^{k,l}_{t_0,t_1} \tag{10}$$

$$\begin{aligned} \text{Volume of domain} = \int_{D^k_{t_0,t_1}} dV &= \sum_l \int_{D^{k,l}_{t_0,t_1}} dV \\ &= \sum_l c^{k,l} = \text{constant} \end{aligned} \tag{11}$$

The temporal scale associated with the domain $D^k_{t_0,t_1}$ is given by:

$$\sigma_{k,t_1,t_2} = \sqrt{\frac{|\ddot{D}^k_{t_0,t_1}|}{|\dot{D}^k_{t_0,t_1}|}} \tag{12}$$

3 Fractals as mathematical tools for Morphogenesis

$$f^m_{k,t_0,t_1} = \sigma_{k,t_1,t_2} \mathcal{X}_{D^k_{t_0,t_1}} \vec{\phi}^m_{k,t_0,t_1} \tag{13}$$

Where $m \in \Lambda_n$, k is a topological index which indexes the different spatial domains that the geometric morphogenetic field is defined over, and $\vec{\phi}^m_{k,t_0,t_1}$ is a component of the geometric morphogenetic potential. This model treats the morphogenetic potential as having two components, the temporal evolution of the potential, and its spatial distribution. We can now extend this definition to n dimensions by taking the composition of n such geometric morphogenetic fields. We take the spatial and temporal dimensions of the morphogenetic field as one dimension of the metric and a different dimension of the time-space. If $m \in \Lambda_n$ is the index for space, and $j \in \Lambda_{n+1}$ for time-space, the j'th component of f at (x_t^k, t) can be expressed as follows:

$$f_{k,t}^m(x_t^k) = \sigma_{(k,t)}\left(f_{k,t}^m(x_t^k), x_{t-}^k, x_{t+}^k\right)$$

To allow for inhomogeneities in the morphogenesis process, we also extend f to a function that takes in a continuous parameter s from the parameter set $\mathcal{S}$, which then acts as a gradient field:

$$f_{k,t}^m : (a, \sigma_{(k,t)}) \longrightarrow f_{k,t}^m(a, \sigma_{(k,t)}) \tag{14}$$

Examples:

3.1 Inhomogeneity in Geometry

This is equivalent to taking M as the boundary of spacetime. The external forcings are then the initial and final states of the spacetime boundaries:

$$f^m(s) = (x_{s(t)}(a), \sigma_t^m(a))$$

With the following boundary conditions

$$x_t(0) \overset{!}{=} \mathbf{p}_\sigma, x_{t'}(1) \overset{!}{=} \mathbf{p}_\sigma, \text{where } t' \leq t$$

This can be extended to the case of a collection of geometric morphogenetic fields with a total of #directions $+ 1$ as follows:

$$f_{t_0}^{m_0 \cdots m_{n-1}} = (x_{t_0}^i(a), \sigma_{k,i,t}^m)$$

with the following boundary conditions

$$x_t^i(0) \overset{!}{=} \mathbf{p}_{\sigma,0}^i, x_{t'}^i(1) \overset{!}{=} \mathbf{p}_{\sigma,n}^i, \text{where } t' \leq t$$

3.2 Self-Organized Criticality in Geometry

This is equivalent to taking M as the state space of a discrete dynamical system whose phase-space is a subset of Euclidean space with non-zero measure E and a boundary of spacetime. The external forcings are then a single point state in the boundary of spacetime:

$$f^m(s) = (x_{s(t)}(a), \mathbf{p}_{i,t}(a), a, \text{ a single point } x_t = (x_{x_1}, \ldots, x_{x_d}, 1 - (1-\alpha)a), \text{where } \alpha = \frac{E}{\Omega}$$

The boundary conditions are then trivial:

$$f^m(0) = (x_{s(0)}(a), \mathbf{p}_{i,0}(a), a, \mathbf{p}_{i,t}(a)), \text{ where } t = 1$$

3.3 Lottery Model

In the lottery model, the temporal evolution of the morphogenetic field is trivial:

$$f_{k,t}^m = x_t^m$$

The evolution of the geometric morphogenetic fields is then given by the initial conditions of the temporal evolution field:

$$f_{k,t}^m = f^m(x_{t-1}^k)$$

3.4 Advected Sandpile

In the advected sandpile, the geometric morphogenetic fields are coupled with a dissipation field and given as:

$$\partial_t \sigma_{i,t,k} + (\vec{\nabla} \cdot f)_{i,t,k} + (\vec{\nabla} \cdot h)_{i,t,k} = 0$$

Where f is the geometric morphogenetic potential, and h is a dissipation potential. This means that we can re-scale f and h by a constant factor Ω to model the fact that they are both proportional to the rate of change of the fractal geometry of the evolving sandpile:

$$\partial_t \sigma_{i,t,k} + (\vec{\nabla} \cdot f)_{i,t,k} + (\vec{\nabla} \cdot h)_{i,t,k} = 0.$$

Where k denotes the fractal dimension of the evolving sandpile.

The dissipation field is written as $\omega : R \to \mathbb{R}$, such that $\omega(y) = 1 + \frac{1}{2\pi i}\Omega(y)$ with $|\Omega(y)| \leq 1$. The dissipation is added to the advected sandpile to help keep the sandpile from collapsing because when it collapses, it will destroy the advective flux of morphogenetic potential through the sandpile.

3.5 Dynamic Evolution

It is now possible to write the dynamic evolution of the advected sandpile:

$$\vec{a} = \vec{a}_0 + \vec{\dot{\vec{a}}}_0$$

$$\dot{a} = \vec{a}_0 - \vec{\dot{\vec{a}}}_0$$

$$\vec{S} = -\vec{S}_0$$

$$S = f + h + (\vec{\dot{\vec{a}}}) + (\vec{\vec{a}}) + (\vec{S} - S_0)$$

$$S_0 = S_0 + \vec{\dot{\vec{S}}}_0$$

$$S = S_0 + (f + h) + (\vec{\dot{\vec{S}}} - S_0)$$

$$S = S_0 + (S_0 + (f + h) + (\vec{\dot{\vec{S}}} - S_0))$$

It is worth noting that, if $\vec{\vec{a}} \neq 0$ (i.e., if the advected sandpile does not collapse), then the above evolution equation will have a solution if and only if $\Omega(x) \leq 1$, where $x = (x, y)$, namely, if and only if

$$\text{and } \exists \epsilon > 0 (x, y) \in \mathbb{R}^{\not\models} \implies |\Omega(x) - 1| \leq \epsilon.$$

The first part of the above equation states that the sandpile has to expand away from any point in the domain (otherwise it will collapse) and then, under the dissipative term, the sandpile will shrink away from any other point and, finally, converge to any point on its own. These points are defined by

$$\{(x, y) : 0 \leq x \leq \lambda, y = f\vec{\vec{a}} - h\}.$$

This is the domain where the problem is expected to be most relevant and, as such, where it has to be solved by using the appropriate numerical scheme to solve the Navier-Stokes equation. Finally, the dissipative term is a conservative, linear, diffusion term that attempts to make the solution converge to the equilibrium state given by the initial conditions. It is worth mentioning that, in this case, the dissipation field is a random force applied on the advected sandpile and that the size of the sandpile is a random quantity with probability density given by

$$\rho(r) = \exp\left(-\int_0^r \delta(r)\,\mathrm{d}r\right).$$

This has to be compared to the case of the advected sandpile:

$$\vec{a}_0 \approx \exp\left(-\int_0^\lambda \delta(\lambda)\,\mathrm{d}\lambda\right)$$

This shows the advantages of using a random force instead of the dissipative term.

3.6 Numerical Implementation

The numerical schemes developed in this paper are written using the advection and diffusion method, which allows the solution of the equation in time to be efficiently calculated, as it only requires knowledge of the flow velocity in the region around the advected sandpile at time t_0 (so long as the solution of the advection equation is known). In this way, the numerical scheme is written using only vector and scalar fields. As an example, to write a linear scheme for the scalar advected sandpile, an explicit method for the evolution equation (1) is written in vector field representation as follows:

$$\vec{t} = \vec{s}_0 - \gamma \vec{H}_0.$$

For the linear scheme, we require the initial condition in vector field form, which can be written as:

$$\vec{0} = \alpha \vec{B}_0 + \beta \vec{A}_0.$$

Here we have written the vector field $\vec{B}_0$ (and correspondingly $\vec{A}_0$) as a linear combination of scalar fields, whose weights are related to the corresponding initial conditions. Using this linearization, the advection equations (1) can be written in the following matrix form:

$$\begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 & \beta \\ 0 & \alpha & \beta \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 & \beta & \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \beta & \\ 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} \alpha & 0 & \beta & \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \beta & \\ 0 & 0 & 0 & \end{bmatrix}$$

Using this matrix, the solution of the advection equation in time can be written as a single matrix-vector product with that matrix, giving:

$$\vec{t} = \vec{s}_0 - \gamma \vec{H}_0 - \alpha \vec{B}_0 + \beta \vec{A}_0 - \alpha \vec{B}_0 + \beta \vec{A}_0 - \gamma \vec{A}_0 - \beta \vec{A}_0 + \gamma \vec{A}_0 - \beta \vec{A}_0 - \gamma \vec{A}_0 + \beta \vec{A}_0 + \gamma \vec{A}_0$$

We note that the solution of this equation is also the solution of the diffusion equation that arises when the sandpile is advected by a stream of fluid (with the same density distribution as the fluid and with no other specific properties; in other words, the sandpile moves according to a simple diffusion process). Similarly, the solution of the diffusion equation is also the solution of the equation (1) that would arise when the sandpile is advected by a stream of fluid (with the same density distribution as the fluid and with no other specific properties; in other words, the sandpile moves according to a simple diffusion process).

When the flow is laminar, we have that the matrix $\vec{B}_0$ vanishes, and we can simplify (1) by eliminating $\vec{A}$:

$$\vec{t} = \gamma\,\vec{s}_0 - \beta\,\vec{A}_0 + \gamma\,\vec{A}_0 - \beta\,\vec{A}_0 + \gamma\,\vec{A}_0 + \beta\,\vec{A}_0 + \gamma\,\vec{A}_0$$

We find that, in this case, the solution of the diffusion equation becomes (for the sake of brevity):

$$\vec{s}(t) = \vec{A}(t) - \gamma\,\vec{A}_0 + \beta\,\vec{A}_0$$

where, by the third equation in (2), $\vec{A}(t)$ can be written as a linear combination of two independent functions (corresponding to the linear diffusion processes in the x-and y-directions):

$$\vec{A}(t) = A_0 + A_1\exp(\frac{\pi}{2}t) + A_2\exp(\frac{\pi}{2}t)$$

The solution of (2) will then be the same solution of (1) that would arise when the sandpile is advected by a stream of fluid of density A_0 (with no other specific properties) that moves according to the simple diffusion process that arises when the sandpile is advected by a stream of fluid of density A_0 (with no other specific properties). However, in this case, the value of A_0 is constant and equal to $1/m$ (where m is the mass of sand per unit area that is on the advected sandpile). Let us consider the case that the advected sandpile is initially flat. This situation is particularly interesting, because the solution of the problem (1) will show the behavior of the sandpile. In this case, we can use the definition of the stress tensor

$$\sigma = (\vec{A}^\dagger - \vec{B}^\dagger)\partial_v \otimes \vec{A}$$

and the definition of the stress tensor in the x-direction (that is, for the flow in the x-direction and for the x-direction alone):

$$\sigma_{xx} = 0$$

We find that, in this case, the stress tensor is:

$$\sigma_{xx} = \frac{A_0\sigma_x^\dagger + \vec{B}_0\sigma_x}{-}\frac{A_0\sigma_x^\dagger - A_0\sigma_x}{\sum_{x=1}^{\infty}\exp(\frac{\pi}{2m)} - \vec{B}_0\sigma_x}(\vec{A}^\dagger - \vec{B}^\dagger)\partial_v + \frac{A_0\sigma_x^\dagger - A_0\sigma_x}{\sum_{x=1}^{\infty}\exp(\frac{\pi}{2m)} - \vec{B}_0\sigma_x}$$

However, we can show that the stress tensor has the following solution (it is not unique, but a finite set of solutions):

$$\sigma_{xx} = \frac{A_0\sigma_x^\dagger + \vec{B}_0\sigma_x}{-}\frac{\vec{B}_0\sigma_x}{2} + \frac{\vec{A}_0\sigma_x}{2} - \frac{1}{(\vec{A}^\dagger - \vec{B}^\dagger)\partial_v}$$

Since $\vec{A}_0$ has a finite norm (because it is bounded, hence positive), this is a solution of the problem. We also have that this solution satisfies the following relations

$$\frac{\partial\sigma_{xx}}{\partial x} = -\frac{A_0}{A_0}, \quad \sigma_{xx}(x,0) = \sigma_x(x,0),$$

and

$$\sigma_{xx}(x,\pm\infty) = 0.$$

As a result, we find that the stress tensor σ is finite everywhere (with a particular behavior near the sandpile boundary), but it can be decomposed in two components where the latter is a function of the shape of the sandpile and of the x-direction velocity. This is clearly a very intriguing problem to be analyzed. There are many interesting questions left unanswered, like for instance the behavior of the stress tensor near the x-direction boundary, the flow induced by the sandpile, the formation of vortices, the influence of the advected sandpile on a gravity wave, etc. The solution that we have found for the stress tensor describes a situation in which the sandpile motion is forced to do horizontal water flows (with zero horizontal velocities) due to its own gravity and to a boundary force that acts as a sink of pressure (of the same order of magnitude as the sandpile density). Such a situation can be described by a parabolic equation (σ can be determined from the pressure drop and the horizontal velocity in the problem), and therefore it can be used to study the effect of the sandpile motion on the gravity wave field.

4 Conclusions and Discussion

Morphogenesis is an emergent phenomena that can only be described by the mathematics of nonlinear dynamics. This emergent phenomenon arises because of the discrete nature of spacetime and the continuous nature of physics. So far, all quantum gravity and most theoretical approaches to modeling quantum gravity use a differential geometry, i.e., one where spacetime is an infinitely differentiable manifold. In these theories, the natural boundary conditions that should be used to find classical or semi-classical dynamics is that spacetime is asymptotically flat or anti-de-Sitter. However, these theories leave out the notion that the actual metric might take on some of these geometries, instead it is a constraint. This approach ignores the fact that the set of all of the possible geometries of the universe is an infinite set [**14**], so many of those geometries will give you a divergent (i.e., singular) metric. This divergence is due to the fact that the metric of spacetime is locally Minkowskian in the smooth limit, but the overall structure of the spacetime geometry is locally fractal. This implies that all known theories of gravity, from general relativity, to string theory, to loop quantum gravity are local approximations of an underlying non-linear theory of gravity. This theory of gravity describes the dynamics of spacetime as a fractal system with a hierarchical set of symmetries, i.e., diffeomorphisms, i.e., coordinate transformations.

A Fractal Dimension in the Anatomy of the Brain

There have been attempts to apply fractal geometry to brain anatomy and function [**4**; **2**; **3**; **6**]. The fractal dimension of a geometric object is an exponent of the object's box-counting fractal dimension, an approximation to its Hausdorff dimension. For example, the fractal dimension of a spheroid in 3-D is approximated by the exponent of the number of (nonoverlapping) spherical shells that fit inside its bounding box. So, an estimate of the (approximate) fractal dimension of a brain can be found by calculating the fractal dimension of the cortex as a whole using some appropriate mathematical framework for brain anatomy, and then finding the power law relating this "apparent" fractal dimension and the number of layers in the cortex. Fractal dimension then provides a dimensionless and continuous (though approximate) measure of brain structure. The idea may prove fruitful since it has recently been suggested that cortical dynamics might be fractal in nature [**5**].